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Investigating brane resolution with perturbative dynamics

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Abstract: We investigate the transition between singular and non-singular geometries from the vantage point of perturbative field dynamics. In particular, we obtain the closed-form absorption probability for minimally-coupled massless scalars propagating in the background of a heterotic 5-brane on a Taub-NUT instanton. This is an exact calculation for arbitrary incident frequencies. For the singular geometry, the absorption probability vanishes when the frequency is below a certain threshold, and for the non-singular case it vanishes for all frequencies. We discuss the connection between this phenomenon and the behavior of geodesics in this background. We also obtain exact quasinormal modes.

Keywords: p-branes, AdS/CFT Correspondence, Black Holes.

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Contents

1. Introduction

In certain cases, deformations of the standard brane solutions can have the effect of "resolving" singularities. Such deformations are the result of an additional flux on a Ricci-flat space transverse to the brane. Since this resolution can break additional supersymmetry, such non-singular solutions may serve as viable gravity duals of strongly-coupled Yang-Mills field theories with less than maximal supersymmetry [1]–[6].

In this paper, we look at the deformed heterotic 5-brane on a Taub-NUT instanton, which preserves half of the original supersymmetry [5]. It is a rather pleasant surprise that wave equation of a minimally-coupled, massless scalar propagating in this background can be solved exactly and in closed-form in terms of hypergeometric functions. The scalar dynamics are similar to that in the background of a two-charge black hole in four dimensions. This enables us to study the deformation to a non-singular spacetime from the point of view of perturbative field dynamics.

While absorption probabilities can yield useful information on supergravity backgrounds, additional motivation for studying absorption by p-branes is the conjectured duality between d-dimensional supergravity on certain spacetimes with corresponding $d-1$-dimensional quantum field theories. While this conjecture has been most studied for the case of supergravity on AdS spacetime [7], such a duality may hold in a more general context.

There are only a few known examples for which the wave equations of a scalar field in a p-brane background is exactly solvable. In the case of the extremal D3-brane [8, 9, 11, 12], as well as the extremal D1-D5 intersection [10], the wave equation can be cast into the form of the modified Mathieu equation. The absorption probability can then only be expressed in terms of power-series expansions of the frequency. For the five-dimensional single-charge black hole and the four-dimensional two-charge black hole the absorption probability can be found in closed-form [13, 14].

This paper is organized as follows. In section 2 we present the wave equation for a minimally-coupled, massless scalar propagating in the background of a heterotic 5-brane on a Taub-NUT instanton. In section 3 we find the conditions for non-zero absorption from the relative phase shift between incident and reflected waves. In section 4 we calculate the absorption probability exactly and in closed-form. In section 5 we analyze the connection between absorption probability and radially-infalling timelike geodesics. In section 6 we calculate the quasinormal modes exactly.

2. Scalar perturbations of heterotic 5-brane on Taub-NUT instanton

We begin by briefly reviewing the deformed Heterotic 5-brane solution found in [5]. The bosonic sector of the ten-dimensional heterotic supergravity consists of the metric, a dilaton, a two-form potential $A_{(2)}$ and the Yang-Mills fields of $E_8 \times E_8$ or SO(32). The corresponding lagrangian is

$$L_{\rm het} = R * \mathbf{1} - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{-\phi} * F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{-\frac{1}{2}\phi} * F^i_{(2)} \wedge F^i_{(2)} , \tag{2.1}$$

where

$$\begin{aligned} F_{(3)} &= dA_{(2)} + \frac{1}{2} A^i_{(1)} \wedge dA^i_{(1)} + \frac{1}{6} f_{ijk} A^i_{(1)} \wedge A^j_{(1)} \wedge A^k_{(1)} , \\ F^i_{(2)} &= dA^i_{(1)} + \frac{1}{2} f^i_{jk} A^j_{(1)} \wedge A^k_{(1)} . \end{aligned} \tag{2.2}$$

The heterotic 5-brane deformed by an abelian U(1) field is

$$\begin{aligned} ds^2_{10} &= H^{-1/4} dx_\mu dx^\nu \eta_{\mu\nu} + H^{3/4} ds^2_4 , \\ e^{-\phi} * F_{(3)} &= d^6x \wedge dH^{-1} , \qquad \phi = \frac{1}{2} \log H , \qquad F_{(2)} = m L_{(2)} , \end{aligned} \tag{2.3}$$

where $L_{(2)}$ is a self-dual harmonic 2-form in the Ricci-flat Kähler transverse metric ds^2_4. For the case of a Taub-NUT instanton,

$$ds^2_4 = \left(\frac{r+a}{r-a}\right) dr^2 + 4a^2 \left(\frac{r-a}{r+a}\right) d\psi^2 + (r^2 - a^2)(d\theta^2 + \sin^2\theta d\phi^2) , \tag{2.4}$$

where the radial coordinate has the range $a \le r \le \infty$. The normalisable solution for $L_{(2)}$ is given by

$$L^2_{(2)} = \frac{4}{(r+a)^4} , \tag{2.5}$$

for which half the original supersymmetry is preserved. The Bianchi identity for $F_{(3)}$ is given by

$$dF_{(3)} = \frac{1}{2} F^i_{(2)} \wedge F^i_{(2)} . \tag{2.6}$$

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Solving for the function H, we obtain

$$H = 1 - \frac{4ab + m^2}{4a(r-a)} + \frac{m^2}{4a(r+a)}\,, \tag{2.7}$$

where b is an integration constant. For the case $b = -m^2/(4a)$, the function H becomes non singular in the entire radial coordinate range [5].

The equation of motion for a minimally-coupled scalar field is

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu \Phi = 0\,. \tag{2.8}$$

Taking

$$\Phi(t, r, \theta, \phi, \psi) = \phi(r) Y_{\ell,k}(\theta,\phi) \mathrm{e}^{in\psi} \mathrm{e}^{-i\omega t}\,, \tag{2.9}$$

we find the radial equation to be

$$\frac{1}{r^2 - a^2}\partial_r (r-a)^2 \partial_r \phi + \left[\omega^2 H - \frac{r+a}{r-a}\frac{n^2}{4a^2} - \frac{\ell(\ell+1)}{r^2-a^2}\right]\phi = 0\,. \tag{2.10}$$

Making the wave function and coordinate transformations $\phi(r) = (r-a)^{-1}\psi(r)$ and $z = r - a$, together with (2.7), the radial wave equation (2.10) becomes

$$\begin{aligned}\partial_z^2\psi + \Bigg[(\omega^2 - \frac{n^2}{4a^2}) + \left(\omega^2\left(2a + \frac{m^2}{4a} - \frac{4ab+m^2}{4a}\right) - \frac{n^2}{a}\right)\frac{1}{z} - \\ - \left(\frac{\omega^2}{2}(4ab + m^2) + n^2 + \ell(\ell+1)\right)\frac{1}{z^2}\Bigg]\psi = 0\,. \end{aligned} \tag{2.11}$$

This equation can be solved exactly in terms of hypergeometric functions. Note that wave-like solutions exist only for $\omega > n/2a$.

3. Phase shifts and absorption

Before solving for the absorption probability, it is instructive to examine the relative phase shift between incident and reflected waves, which is obtained directly from the exact wave function solution. If the phase shift has an imaginary component, then there is a non-zero absorption probability [15].

3.1 Non-singular geometry

In the case of a non-singular background for which $b = -m^2/(4a)$, the wave equation (2.11) is

$$\partial_z^2\psi + \left[\left(\omega^2 - \frac{n^2}{4a^2}\right) + \left((2a - b)\omega^2 - \frac{n^2}{a}\right)\frac{1}{z} - \frac{(n^2 + \ell(\ell+1))}{z^2}\right]\psi = 0\,. \tag{3.1}$$

This wave equation is of the same form as that for a four-dimensional single-charge black hole. It is also formally equivalent to the Schrödinger equation for a particle of energy $E = \omega^2 - n^2/(4a^2)$ in an attractive Coulomb potential of charge $Q = (2a - b)\omega^2 - n^2/a$

with an effective angular momentum given through $L(L+1) = \ell(\ell+1) + n^2$. The exact solution is

$$\psi = N(\sqrt{E}z)^{L+1} \mathrm{e}^{-i\sqrt{E}z} M \left(\frac{iQ}{2\sqrt{E}} + L + 1, 2L + 2, 2i\sqrt{E}z \right), \tag{3.2}$$

where M is Kummer's regular confluent hypergeometric function and the factor N ensures that the solution is Dirac normalized. The asymptotic form for large z is

$$\psi \sim \sin \left(\sqrt{E}z + \frac{Q}{2\sqrt{E}} \log \sqrt{E}z - \frac{L\pi}{2} + \delta \right), \tag{3.3}$$

where

$$\delta = \arg \Gamma \left(L + 1 - i \frac{Q}{2\sqrt{E}} \right). \tag{3.4}$$

The relative phase shift between incident and reflected waves is

$$2\Delta = -L\pi + 2\delta\,. \tag{3.5}$$

Δ is real, which implies that the incoming and outgoing flux at infinity are equal. Thus, the absorption probability is zero at all frequencies, which has been well-established for Coulomb potentials, even those that are attractive [15].

3.2 Singular geometry

The above analysis can be repeated for arbitrary b, which include the case of singular geometry. The wave equation remains formally identical to that of the Schrödinger equation for a particle of energy E in an attractive Coulomb potential. The effective charge is given by

$$Q = \left(2a + \frac{m^2}{4a} - \frac{4ab + m^2}{4a} \right) \omega^2 - n^2/a\,, \tag{3.6}$$

and the effective angular momentum is given through

$$L(L+1) = \ell(\ell+1) + n^2 + (4ab + m^2)\frac{\omega^2}{2}\,. \tag{3.7}$$

L develops an imaginary component for

$$\frac{\epsilon\omega^2}{2} > (\ell + 1/2)^2 + n^2\,, \tag{3.8}$$

where $b = -m^2/(4a) - \epsilon$. Nevertheless, the solution and calculation of the phase shift remains valid for complex L and we find that

$$2\Delta = -L\pi + 2\delta\,, \tag{3.9}$$

where δ is given by (3.4). When L has a non-zero imaginary component, the phase shift is complex and the incoming flux is not equal to the outgoing flux. This indicates that the absorption probability is non zero for $\epsilon\omega^2/2 > (\ell + 1/2)^2 + n^2$. That is, for the singular geometry for which $\epsilon > 0$, there is non-zero absorption above a certain threshold frequency for the incident scalar wave. Note that it is the waves of higher frequency that penetrate through the potential barrier surrounding the singularity. Thus, it is as though nature cloaks the singularity from the outside world via absorption.

4. Closed-form absorption probability

In general, the wave equation of a massless scalar in the background of a p-brane is independent of the world-volume dimension d. This implies that the wave equation is invariant under double-dimensional reduction of the corresponding p-brane. For example, a scalar in the background of an M2-M5 intersection in 11 dimensions and a six-dimensional dyonic string shares the identical wave equation as for the D1-D5 intersection. However, since a two-charge black hole in four dimensions can be obtained from the D1-D5 intersection only via both vertical as well as diagonal reduction steps, the corresponding scalar wave equation changes. The corresponding wave equation is exactly solvable in terms of hypergeometric functions, and the absorption probability can be expressed in closed form. Another interesting limit is when the D1-brane disappears, in which case the wave equation in the D5-brane background is exactly solvable in terms of Bessel functions. This is a second case for which the absorption probability can be obtained in closed form. Dimensionally reducing the D5-brane down to a five-dimensional single-charge black hole does not change the wave equation [13].

The wave equation for a minimally-coupled, massless scalar propagating in the background of a heterotic 5-brane on a Taub-NUT instanton is identical to such a wave equation in the background of a four-dimensional two-charge black hole. To make this apparent, we make the wave function transformation $\psi = z\phi$ and the coordinate transformation $\rho = \sqrt{\omega^2 - n^2/(4a^2)}z$, the latter of which is valid for $\omega > n/(2a)$. The wave equation can be expressed as

$$\partial_\rho^2\phi + \frac{2}{\rho}\partial_\rho\phi + \left[\left(1+\frac{\lambda_+}{\rho}\right)\left(1+\frac{\lambda_-}{\rho}\right) - \frac{L(L+1)}{\rho^2}\right]\phi = 0\,. \tag{4.1}$$

This is of the form of a minimally-coupled scalar field of effective angular momentum given by $L(L+1) = \ell(\ell+1)+n^2$ propagating in the background of a four-dimensional black hole with two charges given by

$$\lambda_\pm = \frac{(2a-b)\omega^2 - n^2/a}{\sqrt{4\omega^2 - n^2/(a^2)}} \pm \sqrt{\frac{((2a-b)\omega^2 - n^2/a)^2}{4\omega^2 - n^2/(a^2)} + (4ab+m^2)\frac{\omega^2}{2}}\,. \tag{4.2}$$

From above, we can see that the scalar perturbative dynamics in the background of the non-singular geometry $\epsilon = 0$ are equivalent to that in the background of a four-dimensional single-charge black hole, for which it has been shown that the absorption probability is zero [13].

The wave function can be solved exactly with hypergeometric functions yielding

$$\begin{aligned}\phi = \rho^{(iq-1)/2}\mathrm{e}^{-i\rho}\Bigg(&\alpha U\left(\frac{1}{2}+ip+\frac{i}{2}q, 1+iq, 2i\rho\right) + \\ &+\beta M\left(\frac{1}{2}+ip+\frac{i}{2}q, 1+iq, 2i\rho\right)\Bigg)\,,\end{aligned} \tag{4.3}$$

where U and M are Kummer's irregular and regular confluent hypergeometric functions, respectively. We have defined $p \equiv \frac{1}{2}(\lambda_+ + \lambda_-)$ and $q \equiv \sqrt{4\lambda_+\lambda_- - (2L+1)^2}$ as in [13].

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From the asymptotic behavior of this solution, it can be seen that in order to have wave-like behavior near the brane as well as asymptotically far, we require that

$$2a\epsilon\omega^2 > \left(\ell + \frac{1}{2}\right)^2 + n^2 \tag{4.4}$$

and

$$\omega > \frac{n}{(2a)}\,, \tag{4.5}$$

respectively.

One can calculate the closed-form absorption probability to be

$$P = \frac{1 - \mathrm{e}^{-2\pi q}}{1 + \mathrm{e}^{-\pi(q+2p)}}\,, \tag{4.6}$$

if the conditions (4.4) and (4.5) are satisfied; otherwise, $P = 0$. This calculation has already been done explicitly in [13] for the analogous case of the two-charge black hole.

5. Geodesics and absorption

In the case of the extremal $D = 5$ single-charge and $D = 4$ two-charge black holes, it has already been shown that the vanishing of the absorption probabilities below certain threshold frequencies is related to the behavior of geodesics in these backgrounds [13]. That is, if a timelike or null geodesic reaches $r = a$ in finite coordinate time, $P = 0$ for all frequencies. If it takes a logarithmically-divergent coordinate time, $P > 0$ for $\omega > \omega_0$, where ω_0 is some finite frequency. If it takes a power-law-divergent coordinate time, $P > 0$ for $\omega > 0$. In this paper, we discuss examples of the first two cases.

Consider radially-infalling timelike geodesics in the background of a heterotic 5-brane on a Taub-NUT instanton. These are described by the lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -H^{-1/4}\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{r+a}{r-a}\right) H^{3/4} \left(\frac{dr}{d\tau}\right)^2 . \tag{5.1}$$

The equation of motion for t yields

$$\frac{dt}{d\tau} = E H^{1/4}, \tag{5.2}$$

where E is a constant of integration. $L = -1$ for a timelike geodesic, which yields

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 \left(\frac{r-a}{r+a}\right) H^{-1/2} - \left(\frac{r-a}{r+a}\right) H^{-3/4} . \tag{5.3}$$

For the non-singular case $\epsilon = 0$, as we approach $r = a$, $H \sim$ constant. Thus,

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} \sim -(r-a)^{1/2}. \tag{5.4}$$

We find that timelike or null geodesics reach $r = a$ in a finite coordinate time.

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For the singular case of non-vanishing ϵ, near $r = a$ we have $H \sim (r-a)^{-1}$. Thus,

$$\frac{dr}{dt} \sim -(r-a)\,. \tag{5.5}$$

We find that the time taken for timelike or null geodesics to reach $r = a$ is logarithmically divergent. Thus, our results are in agreement with the correpondence between the behavior of infalling geodesics and absorption. This correspondence may be a useful tool to study the absorption of fields in singular/non-singular backgrounds for which we are unable to calculate absorption probabilities analytically.

6. Quasi-normal modes

Note that the wave equation (2.11) is of the form

$$\partial_z^2\psi + \left(A + \frac{B}{z} - \frac{C}{z^2}\right)\psi = 0\,. \tag{6.1}$$

The conditions for wave-like solutions (purely real ω), (4.4) and (4.5), can be expressed as $A > 0$ and $C < -1/4$. These conditions impose the constraint that

$$B > \frac{1}{2a}\left(\frac{m^2n^2}{8a^2} + (\ell+1/2)^2\right). \tag{6.2}$$

However, one way that the wave equation (6.1) differs from those in standard p-brane backgrounds is that B (analogous to the p-brane charge) is not necessarily positive. Thus, by restricting ourselves to solutions of purely real ω we may be missing some interesting dynamical features. In this vein, we compute exactly the quasinormal modes [16, 17].

For large ρ, the wave function solution (4.3) is

$$\phi \sim \rho^{-ip-1}(-A\mathrm{e}^{-i\rho} + \mathrm{e}^{i\rho})\,, \tag{6.3}$$

with A given by [13]

$$A = \frac{\Gamma(\frac{1}{2} + ip + \frac{i}{2}q)(\mathrm{e}^{\pi q} + \mathrm{e}^{-2\pi p})}{2i\Gamma(\frac{1}{2} - ip + \frac{i}{2}q)(2\rho)^{2ip}\mathrm{e}^{-\pi p}\cosh\pi(p - \frac{1}{2}q)}\,. \tag{6.4}$$

A quasinormal mode is a free oscillation of the brane itself, with no incoming radiation driving it. Thus, such modes are defined as solutions which are purely ingoing at the horizon and purely outgoing at infinity.[1] Thus, for quasinormal modes, we require that $A = 0$. This imposes the restriction

$$\frac{1}{2} - ip + \frac{i}{2}q = -N\,, \tag{6.5}$$

where $N = 0, 1, 2, \ldots$. From (6.5), we find the exact quasinormal mode frequencies. For simplicity, we consider the case $n = 0$:

$$\omega_\pm = \frac{-i(2N+1)(2a-b)}{\Delta} \pm \frac{\sqrt{-(2N+1)^2(2a-b)^2 + [(2N+1)^2 - (2\ell+1)^2]\Delta}}{\Delta}\,, \tag{6.6}$$

where $\Delta \equiv (2a+b)^2 + 2m^2 > 0$.

[1] For asymptotically AdS spacetimes, quasinormal modes must vanish at infinity

In order for the geometry to remain stable in the presence of quasinormal modes, we must have $\mathrm{Im}\,\omega < 0$. Note that $\mathrm{Im}\,\omega$ describes the decay of the scalar perturbation. This offers a prediction of the timescale for return to equilibrium of the dual quantum field theory, if one exists [16].

From the above equation for $\omega_\pm$, we see that the geometry is stable against such perturbations if $2a - b > 0$, which implies that the effective charge $B > 0$ in (6.1) (we consider modes such that the term in the square root of (6.6) is positive, so that $\mathrm{Re}\,\omega > 0$). Note that, for the non-singular geometry where $b = -m^2/(4a)$, ω becomes completely imaginary.

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